ON A PROBLEM ON THE IMPULSE CONTACT OF MOTIONS

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We consider a differential game [1-5] with termination on a set $M[x_1 = 0; |x_2| \leqslant \leqslant \mu]$, in which the payoff is the time of hitting onto M. The velocity x_2 of the change in the coordinate x_1 obeys the equation

$$x_2^{\cdot} = \varphi_2(x_1, \ldots, x_{n-1}, v) + u$$

The first (minimizing) player's control u is subject to the impulse constraint

$$\mu^{\circ} - \int_0^{\tau} |u|\, dt = \mu(\tau) \geqslant 0$$

and does not enter into the equations describing the variations of the variables $x_3, \ldots, x_{n-1}$. The second (maximizing) player's control v is an $(n-2)$-vector and the second player chooses it from a certain set Q. A number of theorems have been formulated, permitting us to find the controls and the time of the minimax problem or to form the second player's control enabling him to evade falling onto set M under any action by the first player. We consider three examples.

1. Let the equations of motion have the form

$$\begin{aligned} &x_1^{\cdot} = x_2 = \varphi_1(w), \qquad x_2^{\cdot} = \varphi_2(w, v) + u \\ &x_i^{\cdot} = \varphi_i(w, v) \quad (i = 3, \ldots, n-1) \\ &\mu^{\cdot} = -|u|, \qquad \mu \geqslant 0, \qquad v \in Q(w) \\ &w = [x_1, \ldots, x_{n-1}, \mu], \qquad v = [v_2, \ldots, v_{n-1}] \end{aligned} \tag{1.1}$$

The first player's control u is subject to the constraint

$$\mu^0 - \int_0^{\tau} |u|\, dt = \mu(\tau) \geqslant 0 \tag{1.2}$$

while the second player's control v belongs to a certain closed bounded convex set $Q(w)$ defined for all values of w. Constraint (1.2) admits of impulse jumps in the variables x_2, μ in accord with the formulas

$$x_2^{(1)}(\tau) = x_2(\tau) = x_2(\tau - 0) + \mu_1, \qquad \mu^{(1)}(\tau) = \mu(\tau) = \mu(\tau - 0) - |\mu_1| \tag{1.3}$$

The vector

$$w(\tau) = [x_1(\tau), x_2(\tau - 0), \ldots, x_{n-1}(\tau), \mu(\tau - 0)]$$

is called the position of the game at the instant $t = \tau$. If in position $w(\tau)$ the first player realizes jumps (1.3), then the vector

$$w^{(1)}(\tau) = [x_1(\tau),\ x_2^{(1)}(\tau) = x_2(\tau),\ \ldots,\ x_{n-1}(\tau);\ \mu^{(1)}(\tau) = \mu(\tau)]$$

is not a position.

Suppose that for $t > \tau$ the players realize the finite controls $u(w, v)$, $v(w)$, then the subsequent motion by virtue of system (1.1) takes place just as it would take place if the vector $w^{(1)}(\tau)$ were the position at time $t = \tau$. In what follows the words "motion issues from the position $w^{(1)}(\tau)$" should be understood in precisely this sense. The pair of controls $u(w, v)$, $v(w)$ and the trajectory $w(t > 0, \{u(w, v), v(w)\}, w(t = 0))$ realized by them are said to be admissible if the equality

$$w(0, \{u(w, v), v(w)\}, \quad w(t = 0)) = w^{(1)}(0) = w^{(1)}(t = 0)$$

is fulfilled for $t = 0$ and, furthermore, for all $t > 0$ the trajectory is right-continuous in t has a finite number of jumps compatible with (1.3), satisfies (1.2), and satisfies system (1.1) for almost all $t > 0$.

The problems formulated below are solved by admissible controls and trajectories. In order that the initial value $w(0)$ be formally a position, we ascribe to it a previous history $w(t \leqslant 0)$ $(\varepsilon \leqslant t \leqslant 0)$ which is continuous at $t = 0$. The first player can instantly bring about the transition from the set $M\ [x_1 = 0;\ |x_2| \leqslant \mu]$ to the set $K\ [x_1 = x_2 = 0]$ by a "soft" contact with respect to the coordinate x_1 and the velocity x_2. As a consequence of this we can consider M as the set of game termination.

We formulate the fundamental problems:

Problem 1. Find controls $u^\circ(w, v)$, $v^\circ(w)$ such that the time $T[u, v]$ at which trajectory $w(t, \{u, v\}, w(0))$ first hits on M would satisfy the bounds

$$T[u^\circ, v] \leqslant T[u^\circ, v^\circ] \leqslant T[u, v^\circ]$$

The collection of positions admitting of a solution of Problem 1 is denoted $W_0(w)$.

Problem 2. (Evasion). Find $v_0(w)$ such that the trajectory $w(t, \{u, v_0(w)\}, w(0))$ does not hit onto M for any u and $t > 0$. The existence conditions for $v_0(w)$ delineate a set $W_0(w)$.

Problem 3. (Capture). Find $u_{(3)}(w, v)$ leading the trajectory $w(t, \{u_{(3)}, v\}, w(0))$ onto set M in finite time for any $v(w)$. The existence conditions for $u_{(3)}(w, v)$ generate a set $W_{(3)}(w)$.

Problem 4. (Local variant of Problem 1). Find $u_{(4)}(w, v)$, $v_{(4)}(w)$ and a set $W_{(4)}(w)$ such that the bound

$$T[u_{(4)}, v] \leqslant T[u_{(4)}, v_{(4)}] \qquad (1.4)$$

is fulfilled for any $w(0) \in W_{(4)}(w)$ and any v and such that the bound

$$T[u_{(4)}, v_{(4)}] \leqslant T[u, v_{(4)}] \qquad (1.5)$$

is fulfilled for any pair u, $v_{(4)}$ retaining the trajectory in the region $W_{(4)}(w)$.

2. We restrict ourselves to examining only those systems for which the inclusion

$$M_1\ [x_1 = 0;\ |x_2| > \mu] \subseteq W_0(w) \qquad (2.1)$$

is valid and the whole examination is carried out in the region

$$D^{(1)}(w) = D^{(0)}[x_1 = 0,\ x_2 \leqslant 0] \cup C^{(1)}[x_1 > 0] \qquad (2.2)$$

in which any position can be located by a change in axes direction.

Theorem 2.1. For $w \in D^{(1)}(w)$ let there exist a control $v^{(1)}(w)$ and a function $F^{(1)}(w)$ possessing the following properties:

2.1.1. The function $F^{(1)}(w) = x_2$ when $w \in D^{(0)}$.

2.1.2. The sum $\mu + F^{(1)}(w)$ does not grow along any trajectory $w(t, \{u, v^{(1)}(w)\}, w(0))$, $w(0) \in D^{(1)}$. The inclusions

$$v^{(1)}(w) \in v_0(w), \quad D^{(2)}(w) = D^{(1)} \cap [\mu + F^{(1)}(w) < 0] \in W_0(w) \quad (2.3)$$

are valid under conditions 2.1.1., 2.1.2., i.e., the control $v^{(1)}(w)$ solves Problem 2 in the region $D^{(2)}(w)$.

Proof. Let $w(0) \in D^{(2)}$. The trajectory $w(t, \{u, v^{(1)}(w)\}, w(0))$ cannot hit onto M earlier than the instant t_1 at which the inclusion $w(t_1, \{u, v^{(1)}(w)\}, w(0)) \in D^{(0)}$ is first realized, because x_1 can decrease only when $x_2 \leqslant 0$. However, from 2.1.1. and 2.1.2 there follows the inclusion $w(t_1, \{u, v^{(1)}(w)\}, w(0)) \in M_1$. A subsequent hitting onto M is impossible in accordance with condition (2.1).

Under certain conditions the solution of the following problem can serve as the basis for the construction of the functions $v^{(1)}(w)$ and $F^{(1)}(w)$.

Problem 5. Find $v_{(5)}(w)$ minimizing the value of the functional

$$V(w(t, \{0, v(w)\}, w(0))) = x_2(t_1) \ [\text{when } x_1(t_1) = 0]$$

equal to the value of x_2 when the equality $x_1 = 0$ is first realized on the trajectories of system (1.1) obtained for $u(w, v) = 0$.

The collection of values of w for which Problem 5 has a solution forms a set $D_{(5)}(w) \in \in D^{(1)}(w)$. Suppose that we have succeeded in continuing the functions $H_{(5)} = = V(\{v_{(5)}\}), v_{(5)}$ into region $D^{(1)}$ and in obtaining a continuously-differentiable function $F_{(1)}^{(1)}(w)$ and a certain function $v_1^{(1)}(w)$ satisfying the realtions:

$$(F_{(1)}^{(1)}(w))^{\cdot} = \sum_{i=1}^{n-1} F_{(1)}^{(1)[i]}\varphi_i(w, v_1^{(1)}(w)) + F_{(1)}^{(1)[2]}u - F_{(1)}^{(1)[n]}|u| \quad (2.4)$$

$$F_{(1)}^{(1)[i]} = \partial F_{(1)}^{(1)}/\partial x_i \quad (i = 1, \ldots, n-1), \quad F_{(1)}^{(1)[n]} = \partial F_{(1)}^{(1)}/\partial \mu \quad (2.5)$$

$$P_1^{(1)}(w, v_1^{(1)}(w)) = \sum_{i=1}^{n-1} F_{(1)}^{(1)[i]}\varphi_i(w, v_1^{(1)}(w)) \leqslant 0 \quad (2.6)$$

$$|F_{(1)}^{(1)[2]}| \leqslant F_{(1)}^{(1)[n]} + 1 \quad (2.7)$$

Let us explain the relations written out. The left-hand side of (2.4) is the first derivative of the function $F_{(1)}^{(1)}(w)$ by virtue of system (1.1). The function $P_1^{(1)}$ combines the terms not depending on control u, while $P_2^{(1)}$ combines the terms which do depend on control u. In the formulas we have adopted the notation $P_2^{(1)} = F_{(1)}^{(1)[2]}u - F_{(1)}^{(1)[n]}|u|$. It is not difficult to verify the relation

$$(\mu + F_{(1)}^{(1)}(w))^{\cdot} = -|u|(1 + F_{(1)}^{(1)[n]}) + F_{(1)}^{(1)[2]}u + P_1^{(1)} \leqslant 0 \quad (2.8)$$

which is a consequence of bounds (2.6), (2.7). The function $F_{(1)}^{(1)}(w)$ satisfies condition 2.1.1. by construction, while the fulfillment of condition 2.1.2 follows from (2.8). Thus we can assert the following: if the functions $F_{(1)}^{(1)}(w)$ and $v_{(1)}^{(1)}(w)$ satisfy conditions

(2.6) and (2.7), then, by setting $F^{(1)} = F^{(1)}_{(1)}, v^{(1)} = v^{(1)}_{(1)}$, we obtain functions consistent with the hypotheses of Theorem 2.1.

3. Let us assume that in all the subsequent considerations the functions $\varphi_i(w, v)$ $(i = 1, \ldots, n-1)$ and the set $Q(w)$ do not depend upon μ. In this case the functions $H_{(5)} = V(\{v_{(5)}\})$, $v_{(5)}$ and the region $D_{(5)}$ will depend only on the vector

$$s = [x_2, x_1, x_3, \ldots, x_{n-1}] = [x_2, z], \; z = [x_1, x_3, \ldots, x_{n-1}] \tag{3.1}$$

Suppose that we have succeeded in continuing the functions $H_{(5)}$ and $v_{(5)}$ into region $D^{(3)}(s) \supseteq D_{(5)}(s)$, in the form of functions $H^{(3)}$ and $v^{(3)}$,

$$v^{(3)}(s) = v_{(5)}(s), \qquad s \in D_{(5)}(s) \tag{3.2}$$

$$H^{(3)}(s) = H_{(5)}(s), \qquad s \in D_{(5)}(s) \tag{3.3}$$

Let us assume also that from the inclusions

$$s \in D^{(3)}(s), \qquad v \in Q\ (s \in D^{(3)}(s)) \tag{3.4}$$

there follow the relations

$$P_1^{(3)}(s, v^{(3)}(s)) = 0 \leqslant P_1^{(3)}(s, v(s)) \tag{3.5}$$

while from the inclusion $s \in D^{(3)} \cap C^{(1)}$ there follows the bound

$$|H^{(3)[2]}(s)| < 1 \tag{3.6}$$

Here the expression $P_1^{(3)}(s, v)$ equals the derivative of the function $H^{(3)}(s)$ by virtue of system (1.1) with $u(w, v) = 0$.

We introduce into consideration a region $D^{(4)}(w)$ defined in the following manner: for any $w \in D^{(4)}(w)$ there exists an admissible control $u = \mu_1(w)\,\delta \leqslant 0$ such that the vector

$$w^{(1)} = [x_2 + \mu_1, z, \mu - |\mu_1|] = [x_2^{(1)}, z, \mu^{(1)}] \tag{3.7}$$

satisfies the inclusion

$$w^{(1)} \in D^{(3)}(s) \cap [\mu + H^{(3)} \geqslant 0] \tag{3.8}$$

In region $D^{(4)}(w)$ we introduce the impulse control

$$u^{(4)}(w) = \mu_1^{(4)}(w)\,\delta \tag{3.9}$$

where $\mu_1^{(1)}(w)$ is the smallest root of the equation

$$\mu - |\mu_1| + H^{(3)}(x_2 + \mu_1, z) = 0 \tag{3.10}$$

According to conditions (3.6) and (3.8) the smallest root of Eq. (3.9) is unique and nonpositive. The relations

$$\mu + H^{(3)}(s) = 0, \qquad s \in D^{(3)}(s) \tag{3.11}$$

follow from the condition $\mu_1^{(1)}(w) = 0$. Under conditions (3.11) the control $u^{(1)}(w, v)$ is chosen as the smallest root of the equation

$$(\mu + H^{(3)}(s))^{\cdot} = -|u| + H^{(3)[2]}u + P_1^{(3)}(s, v) = 0 \tag{3.12}$$

This root has the form

$$u^{(1)}(w, v) = u^{(4)}(s, v) = -P_1^{(3)}(s, v)\,(1 + H^{(3)[2]})^{-1} \tag{3.13}$$

We present heuristic arguments which suggest how to form the control u in accordance with (3.9) and (3.13). Intuition suggests that the first player should realize with a jump a negative value, maximal in modulus, of the velocity $x_2^{(1)} = x_2 + \mu_1$ with the aim of getting into region $D^{(0)}$ rapidly. However, it is risky for the first player to violate the inequality $\mu^{(1)} + H^{(3)}(x_2^{(1)}, z) \geqslant 0$ by virtue of arguments analogous to those used in the proof of Theorem 2.1. On the other hand, by applying control $u^{(4)}$ the first player is guaranteed from hitting onto set M_1 at least as long as the motion remains in region $D^{(3)}(s)$. Indeed, after the impulse $u^{(4)}(w = \mu_1^{(4)}\delta$ the subsequent motion originates from the position

$$w^{(1)} = [x_2^{(1)} = p^{(4)} = x_2 + \mu_1^{(4)}(w),\ z,\ \mu^{(1)} = \mu - |\mu_1^{(4)}|] \tag{3.14}$$

and takes place in accordance with the equations

$$x_1^{\cdot} = p = x_2, \qquad p^{\cdot} = \varphi_2(p, z, v) + u^{(4)}(p, z, v) \tag{3.15}$$
$$x_i = \varphi_i(p, z, v) \qquad (i = 3, \ldots, n-1)$$

We note that as long as the components of the vector $s' = [p, z]$ remain in the region $D^{(3)}(s') = D^{(3)}(x_2 = p,\ z)$, the equality $\mu + H^{(3)}(s') = 0$ is preserved along the trajectories of system (3.15). This signifies that from the inclusions

$$s'(0 \leqslant t \leqslant t_1,\ \{v(s)\},\ x_2^{(1)}(0) = p(0),\ \ z(0)) \in D^{(3)}(s')$$
$$s'(t_1\{v(s)\},\ p(0),\ z(0)) \in M_2\ [x_1 = 0]$$

(the argument $u^{(4)}(s', v(s'))$ is omitted within the braces because $u^{(4)}$ is a known function of $s', v(s')$) there follows the inclusion

$$w(t_1,\ \{u^{(4)}(w, v(s)),\ v(s)\},\ w(0)) \in M$$

Thus, any control $v(s')$ which retains a trajectory of system (3.15) in region $D^{(3)}(s')$ and brings it at the instant t_1 onto the "plane" $x_1 = 0$, leads the position w onto the set M at this instant t_1.

After making the substitution $p = x_2 + \mu_1$ and taking the condition $\mu_1^{(4)} \leqslant 0$ into account, Eq. (3.10) can be written in the equivalent form

$$\mu - x_2 + p + H^{(3)}(p,\ z) = 0 \tag{3.16}$$

Its solution

$$p^{(4)} = p^{(4)}(\mu - x_2, z) \tag{3.17}$$

jointly with the identity transformation $z' = z$ maps the region $D^{(4)}(w)$ onto the region $D^{(3)}(s')$ and furnishes the initial conditions in system (3.15). In what follows we denote this mapping by $\eta(w)$.

4. Suppose that the first player realizes the control $u^{(4)}$; then the second player is faced with two problems:

Problem 6. Find $v^{(6)}(s')$ such that the trajectory of system (3.15) either remains in the region $D^{(3)}(s')$, but never leaves the plane $x_1 = 0$, or leaves the region $D^{(3)}(s')$. The existence conditions for $v^{(6)}(s')$ delineate a region $D^{(6)}(s') \in D^{(3)}(s')$.

Problem 7. Find $v^{(7)}(s')$ bringing the trajectory of system (3.15) onto the plane $x_1 = 0$ in maximal time. The existence conditions for $v^{(7)}(s')$ delineate a region $D^{(7)}(s') \in D^{(3)}(s')$.

Naturally, the second player will solve Problem 7 only in case it is impossible for him to solve Problem 6. Hence follows the inclusion $D^{(7)}(s') \in D^{(3)}(s') \setminus D^{(6)}(s')$. In the example presented in Sect. 6 the solution of Problem 6 helps to solve the problem of the complete construction of the set W_0 and of the control $v_0(w)$.

Theorem 4.1. Suppose that the exact equality

$$D^{(7)}(s') = D^{(3)}(s') \setminus D^{(6)}(s')$$

holds and that the region $D^{(7)}(s')$ is closed by the boundary $G^{(7)}(s') \ni M_2$ Also suppose that everywhere in region $D^{(7)}(s')$ there exist a function $v^0_{(7)}(s')$ and a continuous function $T_{(7)}(s')$ satisfying the following requirements:

4.1.1. The function $T_{(7)}(s')$ is continuously differentiable in the region $D^{(7)}(s') \setminus \setminus G^{(7)}(s')$ and satisfies the relation

$$T_{(7)}(s', u^{(4)}, \overset{\circ}{v}_{(7)}(s')) = T^{[1]}_{(7)}p + T^{[2]}_{(7)}(u^{(4)}(s', \overset{\circ}{v}_{(7)}(s')) + \varphi_2(s', \overset{\circ}{v}_{(7)}(s'))) + \\ + \sum_{i=3}^{n-1} T^{[i]}_{(7)}\varphi_i(s', \overset{\circ}{v}_{(7)}(s')) = -1 > T_{(7)}(s', u^{(4)}, v(s')) \tag{4.1}$$

4.1.2. Conditions analogous to conditions (4.1) are fulfilled on the part of the boundary $G^{(7)} \setminus M_2$, which is a smooth surface, under a partial differentiation along the indicated part of the boundary.

4.1.3. The function $T_{(7)}(s') > 0$ when $s' \in D^{(7)} \setminus M_2$ and $T_{(7)}(s') = 0$ when $s' \in M_2$.

4.1.4. No control $v(s')$ whatsoever can carry the trajectory outside of $D^{(7)}(s')$ if $s'(0) \in D^{(7)}(s')$.

Under conditions 4.1.1 – 4.1.4 the region $D^{(7)}(s')$ is the total region of existence of the solution of Problem 7 and the equalities

$$v_{(7)}(s') = v^{(7)}(s'), \qquad T_{(7)}(s') = T[v^{(7)}(s')] \tag{4.2}$$

are valid.

The proof of Theorem 4.1 is not complicated and can be omitted. By $D_{(8)}(w)$ and $D_{(9)}(w)$ we denote the preimages of regions $D^{(6)}(s')$ and $D^{(7)}(s')$, corresponding to the mapping $\eta(w)$. In region $D^{(9)}(w)$ the function $T_{(7)}(s')$ goes over into the function $T_{(9)}(p^{(4)}(\mu - x_2, z)\, z)$. We proceed to formulate the conditions under which the pair of controls $u^{(4)}$, $v^{(7)}$ solves Problem 1, 3, 4.

Theorem 4.2. Let the following conditions be fulfilled:

4.2.1. In region $D_{(9)}(w)$ there exists a certain admissible control $v_{(9)}(w)$, becoming $v^{(7)}(s')$ for $x_2 = p^{(4)}(\mu - x_2, z)$ and maximizing the quantity $P_{(9),1}$ which is defined below.

4.2.2. The bound

$$P_{(9),1}(x_2 = p^{(4)}, z, v^{(7)}(s')) \leqslant P_{(9),1}(x_2 > p^{(4)}, z, v_{(9)}(w)) \tag{4.3}$$

is valid for the sum $P_{(9),1}(w, v)$ which is the derivative of function $T_{(9)}$ by virtue of system (1.1) with $u(w, v) = 0$

4.2.3. The bound

$$T^{[2]}_{(9)} = -T^{[n]}_{(9)} > 0 \tag{4.4}$$

is valid.

4.2.4. In the region $D_{(10)}(w) = D^{(1)} \setminus (D_{(9)} \cup D^{(2)})$ there exists a control $v_{(10)}(w)$

such that under any control $u \neq u^{(4)}$, taking the position outside region $D_{(9)}$ at the point $w_1 \in G_{(9)} \setminus M$ of the boundary $G_{(9)}$ the trajectory $w\,(t \geqslant T_{(9)}(w_1),\ \{u,\ v_{(10)}\ (w)\},\ w_1)$ either does not return to the region $D_{(9)}$ at all or returns to the point $w_2 \in G_{(9)} \setminus M$ with the bound $T_{(9)}\ (w_2) \geqslant T_{(9)}\ (w_1)$ being observed.

4.2.5. In region $D^{(1)}$ there exist functions $F^{(1)},\ v^{(1)}$ satisfying the hypotheses of Theorem 2.1, and moreover, $H^{(3)} = F^{(1)}$ when $s \in D^{(3)}\ (s')$.

The inclusions

$$u^{(4)} \in u^\circ, \qquad v^{(7)} \in v^\circ, \qquad D_{(9)} \in W^\circ \tag{4.5}$$

are valid under conditions 4.2.1 – 4.2.5, i.e., $u^{(4)},\ v^{(7)}$ solve Problem 1. Furthermore, the equality

$$T_{(9)}\ (\mu - x_2,\ z) = T\ [u^\circ,\ v^\circ] \tag{4.6}$$

is valid. The equalities

$$\begin{gathered} u^{(4)} = u_{(4)}, \qquad v^{(7)} = v_{(4)}, \qquad D_{(9)} = W_{(4)} \\ T_{(9)}\,(\mu - x_2, z) = T\ [u_{(4)},\ v_{(4)}] \end{gathered} \tag{4.7}$$

are valid under conditions 4.2.1, 4.2.2, 4.2.3, 4.2.5, i.e., $u^{(4)},\ v^{(7)}$ solve Problem 4 in the region $D_{(9)}$. Furthermore, it is obvious that the control $u_4{}^{(4)} = u_{(3)}$ solves Problem 3 in the region $D_{(9)} \in W_{(3)}$ independently of conditions 4.2.1 – 4.2.5.

P r o o f. The derivative of the function $T_{(9)}$ by virtue of system (1.1) has the form

$$(T_{(9)}\ (\mu - x_2,\ z))^{\boldsymbol\cdot} = P_{(9)1} + T_{(9)}^{[2]}\ (u + |\,u\,|) \tag{4.8}$$

Bound (4.3) shows that for $v = v^{(7)}(s')$ this derivative achieves a minimum for $u = u^{(4)}$, over all controls u preserving the inclusion $w^{(1)} \in D^{(4)}\ (w)$. The realization $u < u^{(4)}$ takes the position into region $D^{(2)}$ according to condition 4.2.5. The realization $u \neq u_{(4)}$, taking the position outside $D^{(4)}$ has no advantage, according to condition 4.2.4. When $u = u^{(4)}$ the second player should select $v = v^{(7)}\ (s')$ in accordance with the solution of Problem 7. The proof of the correspondence with Problem 1 is complete. If we discard condition 4.2.4, then the proof of the correspondence with the solution of Problem 4 for all u which paired with control $v_{(3)}\ (w)$ retains the trajectory in region $D_{(9)}$, is a verbatim repetition of the arguments presented above.

T h e o r e m 4.3. If the set $Q\ (s)$ and the functions $\varphi_i\ (s,\ v)\ (i = 2, \ldots\ n - 1)$ are such that in the region

$$L\,(w, c_i) = D_{(9)}\,(w) \bigcup\, [c_2{}^2 + \ldots + c_{n-1}^2 \leqslant a^2]$$

we can find a continuous and bounded function $N\ (w,\ c_i) \to 0$ as $a^2 \to 0$ uniformly in $w \in D_{(9)}(w)$ and satisfying the bound

$$|\,\Delta^{(2)}\,(\max_v\,(c_2\varphi_2 + \ldots + c_{n-1}\varphi_{n-1})\,| \leqslant N\,(w, c_i)\,|\,\Delta x_2| \tag{4.9}$$

(where $\Delta^{(2)}\,\psi\ (s,\ c_i)$ denotes a partial increment of the function ψ with respect to the variable x_2), then there exists a positive function $\varepsilon\ (x_2 < 0) > 0$ and a region $D\,[x_2 < 0,\ 0 \leqslant x_1 \leqslant \varepsilon\ (x_2)]$ in which bounds (4.3) and (4.4) are valid.

P r o o f. We write out in detail the quantity $P_{(9)1}$

$$P_{(9)1} = T^{[1]}_{(9)} x_2 + T^{[2]}_{(9)} \varphi_2 (s, v_{(9)} (w)) + T^{[n-1]}_{(9)} \varphi_{n-1} (s, v_{(9)} (w)) \tag{4.10}$$

in accord with the formulas

$$T^{[1]}_{(9)} = T^{[1]}_{(7)} + T^{[p]}_{(7)} p^{(4)[1]}, \quad T^{[2]}_{(9)} = T^{[p]}_{(7)} p^{(4)[2]} \tag{4.11}$$
$$T^{[i]}_{(9)} = T^{[i]}_{(7)} + T^{[p]}_{(7)} p^{(4)[i]} \qquad (i = 3, \ldots, n-1)$$

which are a consequence of bound (3.6) guaranteeing the differentiability of the function $p^{(4)}(w)$ and the bound $p^{(4)[2]} > 0$, as well as with the relations

$$T^{[i]}_{(7)} = 0 \qquad (i = 2, \ldots, n-1), \quad s' \in [x_1 = 0;\ p < 0] \tag{4.12}$$
$$T^{[1]}_{(7)} (\partial x_2 > 0) > 0, \qquad s' \in [x_1 = 0;\ p < 0] \tag{4.13}$$
$$T^{[p]}_{(7)} > 0, \qquad s' \in [0 < x_1 \leqslant \varepsilon (x_2);\ p < 0] \tag{4.14}$$

We obtain the following conclusions. The bound (4.14) in combination with the bound $p^{(4)[2]} > 0$ guarantees the fulfillment of bound (4.4). Since the function $p^{(4)}(\mu - x_2, z)$ depends only on $\mu - x_2$, after carrying out partial differentiations of this function the quantity $\mu - x_2$ can be replaced by $p^{(4)}(\mu - x_2, z)$ from Eq. (3.16), and as a consequence the quantities $T^{[i]}_{(9)}$ can be taken as dependent on $p^{(4)}, z$. Thus, a change in x_2, preserving the quantities $\mu - x_2$ and $p^{(4)}$, affects only the functions $\varphi_2, \ldots, \varphi_n$. The relation

$$P_{(9)1} (p^{(4)} = x_2, z, v_{(7)} (s')) \leqslant \max_v P_{(9)1} (x_2 \geqslant p^{(4)}, z, v (w)) \tag{4.15}$$

is valid according to bound (4.9) and to the smallness of the derivatives $T^{[2]}_{(9)}, \ldots, T^{[n-1]}_{(9)}$ because in formula (4.10) the fundamental role is played by the first term. Bound (4.15) implies bound (4.3) as a corollary.

Before proceeding to a consideration of examples, we make several remarks.

1. The inclusion $M_1 \in W_0 (w)$ and the bound (3.6) are fundamental constraints for the constructions proposed. The inclusion $D^{(3)} \supseteq D_{(5)}$ is not essential because it is not used in the proof.

2. As we shall see from the example in Sect. 7, the region $D^{(7)} (s') = D^{(3)}(s') \setminus D^{(6)}(s')$ is not always successfully constructed completely. However, the arguments in Theorem 4.1 nowhere make use of the above-mentioned exact equality and are applicable in any part of region $D_{(9)} (w)$.

3. In the examples presented below it is impossible to hit onto M from the set M_1 when $v_0 (w) = 0$. For the first example this follows from an analysis of the problem, while for the second and third examples, this was proven in [4].

5. Example. With a suitable choice of scales a controlled heavy pendulum with an ideal suspension obeys the equations

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = -\sin x_1 + u + v \tag{5.1}$$
$$\mu^{\cdot} = -|u|, \quad v \in Q (w) \equiv 0$$

In the absence of control v Problem 1 becomes a time optimality problem. When $u = 0$ system (5.1) admits of a first integral

$$H_{(5)} = -\sqrt{1 - \cos x_1 + x_2^2} \tag{5.2}$$

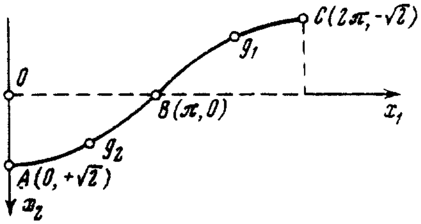

Fig. 1.

corresponding to Problem 5 in the region $D_{(5)}$ located above the separatrix $(A, B, C) \to (1 - \cos x_1 + x_2^2 = 2)$ in the strip determined by the bounds (Fig.1)

$$[0 \leqslant x_1 < 2\pi;\ x_2 \leqslant 0] \cup [0 < x_1 < 2\pi;\ x_2 > 0] \tag{5.3}$$

Let us continue $H_{(5)}$ by formula (5.2) onto the curve (A, B, C) and adopt the following method of mapping the motion. We locate the initial position in the region $D_{(5)}' = D_{(5)} \cup [[A, B, C)]$. We reflect the point $g_1(x_1, x_2) \in [A, B, C)$ with $u(g_1) < 0$ into the point $g_2(2\pi - x_1, - x_2)$ symmetrically with respect to point B, and we assume $u(g_2) = -u(g_1)$. The trajectory is discontinuous under such a transformation, but the function $H_{(5)}$ varies continuously. The region W_0' from which it is impossible to hit onto M is defined by the relation

$$W_0'(w) = [D_{(5)} \cap [\mu + H_{(5)} < 0]] \cup [(A, B, C);\ \mu + H_{(5)}(g) = 0] \\ g \in (A, B, C) \tag{5.4}$$

Indeed, conditions 2.1.1 and 2.1.2 are fulfilled in the region $D^{(2)} = D_{(5)} \cap [\mu + H_{(5)} < 0]$ for the function $F^{(1)} = H_{(5)}$ and Theorem 2.1 is applicable. In the region

$$[(A, B, C);\ \mu + H_{(5)}\ (g \in (A, B, C)) = 0]$$

the motion does not arrive into region $D^{(0)}$ when $u = 0$ while when $u \neq 0$ the motion passes into region $D^{(2)}(w)$. Thus, the motion cannot be led onto M from the region $W_0'(w)$ defined by relation (5.4). The method adopted for representing the motion allows us to identify regions $D^{(3)}$ and $D_{(5)}$ and functions $H^{(3)}$ and $H_{(5)}$ because, obviously, conditions (3.5) and (3.6) are fulfilled. The time $T^{(3)}(x_1, x_2)$ of the point's passing into region $D^{(0)}$ when $u = 0$ has been determined in the region $D^{(3)} \equiv D_{(5)}$.

Under the adopted representation of the motion the function $p^{(4)}(\mu - x_2, z)$ is defined two-valuedly in the region $D^{(4)} = D^{(3)} \cap [\mu + H^{(3)} \geqslant 0]$

$$p_1^{(4)} = {}^1/_2(1 - \cos x_1) / (\mu - x_2) - {}^1/_2(\mu - x_2) \tag{5.5}$$

$$p_2^{(4)} = {}^1/_2(1 - \cos x_1) / (\mu' + x_2') - {}^1/_2(\mu' + x_2') \tag{5.6}$$

Here

$$\mu' = \mu - |\mu_2|, \qquad x_2' = x_2 + \mu_2, \quad \mu_2 > 0 \tag{5.7}$$

$$\mu > \mu_2 = \pm\sqrt{1 + \cos x_1} - x_2 > 0 \tag{5.8}$$

In the last formula the plus sign is taken when $x_1 > \pi$ and the minus, when $x_1 < \pi$. The function $p_2^{(4)}(w)$ is defined in the region delineated by the bound (5.8) and the condition $\mu' > \sqrt{2}$. The introduction of the function $p_2^{(4)}$ has the following geometric meaning. If for $w \in D^{(4)}$ there exists an impulse $u = \mu_4 \delta > 0$ taking the point away from the curve (A,B,C), then it can be represented as a sum of two impulses: the impulse $\mu_2\delta > 0$ leading onto the curve (A,B,C) into the point $g_1(x_1, x_2)$, and the impulse $\mu_3\delta < 0$ transferring the point $g_2(2\pi - x_1, -x_2)$ into the point $(x_1'' = 2\pi - x_1;\ x_2'' = p_2^{(4)}(w))$.

The control

$$u_1^{(4)} = (p_1^{(4)} - x_2)\,\delta = \mu_1^{(4)}\delta \tag{5.9}$$

$$u_2^{(4)} = \mu_2 (g \to g_1)\,\delta + \mu_3 (g_2)\,\delta \tag{5.10}$$

is chosen in accordance with formula (5.9) when the bound

$$T^{(3)}(x_1, p_1^{(4)}) \leqslant T^{(3)}((2\pi - x_1), p_2^{(4)}) \tag{5.11}$$

is realized and with formula (5.10) when the opposite bound is realized. This means that for each position $w \in D^{(4)}$, compatible with bounds (5.8) and condition $\mu' > \sqrt{2}$, there exist two functions

$$T_1^{(4)}(x_1, p_1^{(4)}(\mu - x_2, x_1)), \qquad T_2^{(4)}(-x_1 + 2\pi, p_2^{(4)}(\mu - x_2, x_1))$$

into which the function $T^{(3)}(x_1, x_2)$ transfers after $p_{1,2}^{(4)}$ is substituted for x_2. The bounds

$$T_{1,2}^{(3)[2]} = T^{(3)[p]} p_{1,2}^{(4)[2]} \geqslant 0, \qquad T^{(3)[1]} + T^{(3)[p]} p_{1,2}^{(4)[1]} > 0 \tag{5.12}$$

are valid for both functions [6]. If the control $u_1^{(4)}$ is chosen in accordance with (5.9), then bounds (5.12) guarantee the fulfillment of conditions (4.3) and (4.4), and the control $u_1^{(4)}$ turns out to be optimal in comparison with all controls not leading onto the boundary (A, B, C). On the other hand, the estimate $(T_2^{(3)})^{\bullet} \geqslant -1$, is fulfilled when $u = u_1^{(4)}$ and therefore bound (5.11) is preserved along the trajectory until bound (5.8) or the bound $\mu' > \sqrt{2}$ is violated and the function $T_2^{(3)}$ ceases to exist. The proof of the optimality of $u_2^{(4)}$ when it is chosen in accordance with (5.10) is analogous. Thus, the choice of (5.9) or (5.10) with respect to bound (5.11) or to the bound contrary to it realizes, in system (5.1), a time-optimal hitting onto set M in the region $D^{(4)}(w)$.

6. Example. Suppose that system (1.1) has the form

$$x_1^{\bullet} = x_2, \quad x_2^{\bullet} = u + v, \quad \mu^{\bullet} = -|u|, \quad \mu \geqslant 0, \quad |v| \leqslant 1 = Q(w) \tag{6.1}$$

The controls

$$v^{(1)}(s) = -1, \qquad s \in D'\,[x_1 \geqslant 0;\ x_2 \leqslant 0] \tag{6.2}$$

$$v^{(1)}(s) = 0, \qquad s \in D''\,[x_1 > 0;\ x_2 > 0] \tag{6.3}$$

and the function

$$F_{(s)}^{(1)} = -\sqrt{x_2^2 + 2x_1} \tag{6.4}$$

satisfy (2.2) and (2.3). This means that Theorem 2.1 is fulfilled in the region

$$D^{(2)} = D^{(1)} \cap [\mu + F^{(1)} < 0]$$

and the control $v^{(1)}(s)$ effects an escape.

The region $D^{(3)}(s') = D'$ the control $v^{(3)}(s) = v^{(1)}$ and the function $H^{(3)} = F^{(1)}$ satisfy conditions (3.5) and (3.6). From the inequality $\mu + F^{(1)} \geqslant 0$ and the equation

$$\mu - x_2 + p - \sqrt{p^2 + 2x_1} = 0 \tag{6.5}$$

follows the relation

$$p^{(4)} = x_1 / (\mu - x_2) - \tfrac{1}{2}(\mu - x_2) \leqslant 0 \tag{6.6}$$

proving that the region $D^{(1)} \cap [\mu + F^{(1)} \geqslant 0]$ coincides with the region $D^{(4)}(w)$, since $(p^{(4)}, x_1) \in D^{(3)}(s')$

The control

$$u^{(4)} = (p^{(4)} - x_2)\,\delta \quad \text{for} \quad p^{(4)} - x_2 < 0 \tag{6.7}$$

$$u^{(4)}(s', v) = p(v+1)/(\sqrt{p^2+2x_1} - p) \quad \text{for} \quad p^{(4)} = p = x_2 \tag{6.8}$$

leads to a system (3.15) of the form

$$x_1^{\bullet} = p, \qquad p^{\bullet} = u^{(4)}(s', v) + v \tag{6.9}$$

Setting $v^{(4)}(w) = +1$ when $w \in D^{(4)}$, in the region $D^{(3)}$ we obtain the control $v^{(3)} = +1$, while system (6.9) goes over into the system

$$\psi_1^{\bullet} = 2(\psi_1/\psi_2) - 1 = 2\beta - 1, \qquad \psi_2^{\bullet} = -1 \tag{6.10}$$

$$\psi_{1,2} = \sqrt{p^2 + 2x_2} \pm p \tag{6.11}$$

admitting in $D^{(3)}(s')$ of first integrals of the form

$$\psi_2 + t = c_1, \qquad \psi_2(1 - 3\beta)^{1/3} = c_2 \tag{6.12}$$

After substitution from Eq. (6.5) the function $R^{(3)} = \psi_2(3\beta - 1)$ becomes the function $R^{(4)}(w)$, and the condition $R^{(4)}(w) > 0$ delineates the region $D_{(8)}(w) = D^{(4)} \cap [R^{(4)} > 0]$. Let us compute

$$R^{(3)[1]} = (3\beta - 1)^{-2/3}\sqrt{p^2 + 2x_1} > 0; \quad w \in D_{(8)} \tag{6.13}$$

The equation $(R^{(3)})^{\bullet} = R^{(3)[1]}p + R^{(3)[p]}\beta = 0$ permits us to obtain, from bound (6.13) and the bound $x_2 \geqslant p$ the bound

$$(R^{(4)})^{\bullet} = (R^{(3)[1]} + R^{(3)[p]}p^{(4)[1]})x_2 + R^{(3)[p]}p^{(4)[2]}[|u| + u + 1] \geqslant 0, \quad w \in D_{(8)} \tag{6.14}$$

which is valid for any u preserving the trajectory in region $D^{(4)}$. It ensues from bound (6.13), from the equality $R^{(3)[p]}\beta = -p^{(4)}R^{(3)[1]} \geqslant 0$ for $p^{(4)} \leqslant 0$, and from the bound $p^{(4)[2]} \geqslant 0$. From (6.14) it follows that when $w(0) \in D_{(3)}$ either the inclusion $w(t, \{u, v^{(4)} = +1\}, w(0)) \in D_{(8)} \not\ni M$, is preserved or the trajectory falls into region $D^{(2)}(w)$. This proves the inclusion

$$v^{(4)} = +1 = v_\infty \in v_0; \qquad D_{(8)}(w) \in W_0(w)$$

For $w \in D_{(9)}(w)$, $s' \in D^{(7)}(s') = D^{(3)} \cap [R^{(3)} \geqslant 0]$, Eqs. (6.12) generate the time $T_{(7)}(s')$

$$T_{(7)}(s') = \psi_2(1 - (1 - 3\beta)^{1/3})$$

The function $T_{(7)}(s')$ is continuously differentiable when $1 - 3\beta > 0$ and after the substitution of function (6.5) in the place of p becomes the function $T_{(9)}(w)$. The partial derivatives of these functions have the form

$$T_{(7)}^{[p]} = \psi_2(1 - \lambda)(1 + \lambda - 2\lambda^2)/3\lambda^2 q \geqslant 0 \tag{6.15}$$

$$0 \leqslant \lambda^3 = 1 - 3\beta \leqslant 1, \qquad q = \sqrt{p^2 + 2x_1} \tag{6.16}$$

$$T_{(9)}^{[1]} = (1 - \lambda - 2p/(\psi_2\lambda^2)/q + T_{(7)}^{[p]}/\psi_2 \geqslant 0 \tag{6.17}$$

$$T_{(9)}^{[2]} = T_{(7)}^{[p]}(x_1\psi_2^{-2} + 1/2) \geqslant 0 \tag{6.18}$$

Bound (6.15) is obvious. Bound (6.17) follows from (6.15) and (6.5). Bound (6.18) follows from (6.15).

We can show that the bound $3\beta - 1 \leqslant 0$ is not violated for any v. This fact together with bound (6.15) permits us to apply Theorem 4.1 and to show that the control $v^{(7)} = +1$ solves Problem 7 for system (6.10), since, obviously, the boundary $G[R^{(3)} = 0] \setminus [x_1 = 0]$ also satisfies condition 4.1.2, while the remaining conditions of the theorem follow from formula (6.15). Formulas (6.17), (6.18) attest to the fulfillment of bounds (4.3), (4.4), the condition 4.2.4 is a consequence of the inclusion $D_{(10)} = D_{(8)} \in W_0$, while condition 4.2.5 is obviously fulfilled. Theorem 4.2 is completely applicable in this case and the equalities

$$u^{(4)} = u^\circ, \quad v^{(7)} = +1 = v^\circ, \quad D_{(9)} = W^\circ$$
$$v^{(1)} = -1 = v_0(w), \; w \in D^{(2)}; \; v^{(4)} = v^{(8)} = v_0(w) = +1, \quad w \in D_{(8)}$$
$$D^{(2)} \cup D_{(8)} = W_0, \; W^0 \cup W_0 = D^{(1)}$$

are valid.

7. Example. Let Eqs. (1.1) have the form

$$x_1^\cdot = x_2, \; x_2^\cdot = -x_1 + u + v, \qquad \mu^\cdot = -|u|, \quad \mu \geqslant 0, \; |v| \leqslant 1 = Q(w) \tag{7.1}$$

The controls

$$v^{(3)}(w) = -1, \quad w \in D' \; [x_1 \geqslant 0, x_2 \leqslant 0] \tag{7.2}$$
$$v^{(3)}(w) = +1, \quad w \in D'' \; [x_1 > 0; \; x_2 > 0] \tag{7.3}$$

together with the system

$$x_1^\cdot = x_2, \quad x_2^\cdot = v^{(3)}(w) - x_1, \; \mu^\cdot = 0 \tag{7.4}$$

generate in the region $D^{(1)}$ a first integral of system (7.4), having the form

$$\xi(s) = -\sqrt{(\zeta(x_1 + 1, x_2))^2 - 1}, \quad w \in D' \tag{7.5}$$
$$\xi(s) = -\sqrt{(\zeta(x_1 - 1, x_2) + 2)^2 - 1}, \quad w \in D'' \tag{7.6}$$
$$\zeta(x_1, x_2) = \sqrt{x_1^2 + x_2^2} \tag{7.7}$$

As the position passes from the region D'' to the segment $B[0 < x_1 < 1, \; x_2 = 0]$ the function $\xi(s)$ increases by a jump. It is continuously differentiable at the remaining positions $w \in C^{(1)}$.

Consider the function

$$F_1(w) = \max_{\mu_1} (\mu - |\mu_1| + \xi(x_1, x_2 + \mu_1)) \tag{7.8}$$

the region

$$D^{(2)}(w) = D^{(1)} \cap [F_1(w) < 0] \tag{7.9}$$

and the controls

$$v^{(1)}(w) = -1, \quad w \in D^{(2)} \cap D' \tag{7.10}$$
$$v^{(1)}(w) = +1, \; w \in D^{(2)} \cap D'' \tag{7.11}$$

The inclusions

$$v^{(1)}(w) \in v_0(w), \; D^{(2)} \in W_0(w) \tag{7.12}$$

are valid. The proof of inclusions (7.12) is carried out along a plan analogous to the plan for Theorem 2.1. It is sufficient to establish the equality $\mu - x_2 = F_1(w)$ for $w \in D^{(0)}$ and the fact that when $v = v^{(1)}(w)$ the first player cannot increase the function $F_1(w)$.

Let us discuss the first player's possibilities. It can be verified that when $s \notin B$ the function $H^{(3)}(s) = \xi(s)$ satisfies conditions (3.5), (3.6). As a consequence of the discontinuity of the function $H^{(3)}(p, x_1)$ on the segment B in the region

$$D^{(5)}(w) = B_1[0 < x_1 < 1; \; x_2 \geqslant 0] \cap \{D^{(4)} = D^{(1)} \cap [\mu + H^{(3)}(s) > 0] \tag{7.13}$$

the equation

$$\mu - x_2 + p + H^{(3)}(p, x_1) = 0 \tag{7.14}$$

can have two roots. For $w \in D^{(4)}(w)$ we denote the smallest root of Eq. (7.14) by $p^{(4)}(w)$ and we form the control $u^{(4)}$ by the formulas

$$u^{(4)}(w) = (p^{(4)}(w) - x_2)\delta, \qquad \text{for} \quad p^{(4)}(w) - x_2 < 0 \tag{7.15}$$

$$u^{(4)}(w, v) = -(H^{(3)[1]}p + H^{(3)[2]}(-x_1 + v))(1 + H^{(3)[2]})^{-1} \tag{7.16}$$

$$\text{for} \quad p^{(4)}(w) = x_2, \quad w \in B$$

For $p^{(4)}(w) = x_2$, on the segment $w \in B$ we define the control $u^{(4)}(w, v)$ from formula (7.16), substituting for $H^{(3)[2]}$ the quantities

$$H^{(3)[2]}(x_2 + 0, x_1) \qquad \text{for} \quad -x_1 + v \geqslant 0 \tag{7.17}$$

$$H^{(3)[2]}(x_2 - 0, x_1) \qquad \text{for} \quad -x_1 + v < 0 \tag{7.18}$$

If the control $u^{(4)}(w)$ is formed in accordance with (7.15) - (7.18), then in analogy with the preceding example, the second player's natural reaction is the control

$$v^{(4)} = +1, \quad w \in D^{(4)}(w) \tag{7.19}$$

The pair $[u^{(4)}, v^{(4)}]$ generates a system of equations

$$x_1^{\cdot} = p, \quad p^{\cdot} = -x_1 + 1 + 2p\,(p - H^{(3)}(p, x_1))^{-1}, \quad s' \in D'(s') \tag{7.20}$$

$$x_1^{\cdot} = p, \quad p = 1 - x_1, \quad s' \in D''(s'') \tag{7.21}$$

8. Without proofs, which would take up too much space, we state a number of properties of the motions taking place by virtue of system (7.20), (7.21).

8.1. Among the solutions of system (7.20) with initial conditions on the segment

$$B_2\ [1 < x_1(0) = x_1^\circ \leqslant 2;\ p(0) = 0]$$

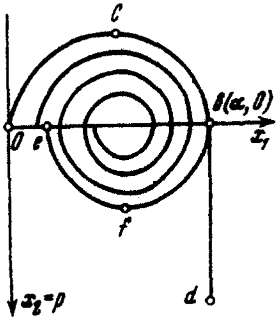

Fig. 2.

there exists a solution $s_\alpha'(\alpha < 2, 0, 0 \leqslant t \leqslant t_1(\alpha))$, and, moreover, only one, possessing the property

$$\lim p(\alpha, 0, t) = \lim x_1(\alpha, 0, t) = 0 \quad \text{for } t \to t_1(\alpha)$$

8.2. All solutions $s'(1 < x_1^\circ < \alpha, 0, t > 0)$ satisfy for some $t_1(x_1^\circ)$ the relations

$$p(x_1^\circ, 0, t_1(x_1^\circ)) = 0, \qquad 0 < x_1(x_1^\circ, 0, t_1(x_1^\circ)) < 1$$

$$s'(1 < x_1^\circ < \alpha, 0, 0 \leqslant t \leqslant t_1(x_1^\circ)) \in D'(s')$$

8.3. The solutions $s'(s'^{\circ} \in B_2, t < 0)$ of system (7.21) follow along the semicircles

$$(1 - x_1)^2 + p^2 = (1 - x_1^\circ)^2, \quad s' \in D''$$

8.4. The solutions

$$s'(1 \leqslant x_1^\circ < \alpha;\ 0,\ 0 \leqslant t \leqslant t_1(x_1^\circ)); \quad s'(1 \leqslant x_1^\circ < \alpha,\ 0 - \pi \leqslant t \leqslant 0)$$

fill the region $B_{(3)}(s')$ bounded by the curves (Fig. 2)

$$s_\alpha'(\alpha, 0, 0 \leqslant t \leqslant t(\alpha)) \to [0, cb]$$

$$s'(\alpha, 0, -\pi \leqslant t \leqslant 0) \to ((x_1 - 1)^2 + p^2 = (1 - \alpha)^2) \to [b, f, e]$$

and the segment

$$B_1'\ [0 < x_1 < \alpha - 1;\ p = 0] \to (0, e)$$

Note that the curves $[0, c, b]$, $[b, f, c]$ do not occur in the region $B_{(3)}(s')$, while the segment $(0, e)$ belongs to it.

8.5. The inclusions

$$v^{(4)}(s') = +1 \in v^{(6)}(s'), \quad B_{(3)}(s') \in D^{(6)}(s')$$

are valid, which signifies that for $u = u^{(4)}$ and for $(p^{(4)}(w), x_1) \in B_{(3)}(s')$ the control $v^{(4)}(s') = +1$ takes the point out of region $D^{(3)}(s')$ into region $D^{(2)}(w)$.

Before we make the assertions to follow, from the point $b(\alpha, 0)$ we draw a vertical halfline $[b, d)$ and we denote the curve $[0cbd)$ by $G^{(7)}(s')$. By $B_{(7)}(s')$ we denote the region located to the right of and above this curve and including it. We assume that the inclusions

$$s' = [p^{(4)}(w), x_1] \in B_{(7)}(s'); \quad w \in D^1_{(7)}(w)$$

are equivalent, i.e., we construct a region $D^1_{(7)}(w)$ which the transformation $\eta(w)$ takes into the region $B_{(7)}(s')$.

8.6. The inclusions

$$u^{(4)}(w, v) \in u_{(3)}(w, v), \qquad D^1_{(7)} \in W_{(3)}(w)$$

are valid. Equations (7.20), (7.21) generate in region $B_{(7)}(s')$ the time

$$T[u^{(4)}(s', v^{(4)}), v^{(4)} = +1] = T_{(B,7)}(s')$$

while in the region $D^{(1)}_{(7)}(w)$ this time becomes the function

$$T^1_{(7)}(\mu - x_2, x_1) T_{(B,7)}(p^{(4)}(\mu - x_2, x_1), x_1)$$

8.7. Let $A_1 > 0$ be the largest of the numbers A such that the bound $T^{[p]}_{(B,7)} \geqslant 0$ follows from the bound $T_{(B,7)} \leqslant A$. The control

$$v_1^{(7)} = v(4) = +1 \qquad \text{for} \quad w \in B^{(7)} = B_{(7)} \cap [T_{(B,7)} \leqslant A_1]$$

solves Problem 7, i.e., the inclusions

$$v_1^{(7)} \in v(7)(s'), \qquad B^{(7)}(s') \in D^{(7)}(s')$$

are valid.

8.8. Let $A_2 > 0$ be the largest of the numbers A such that the bound $T_{(B,7)} \leqslant A$ implies the bound

$$T^{1[1]}_{(7)} = T^{[1]}_{(B,7)} + T^{[p]}_{(B,7)} p^{(4)[1]} \geqslant 0$$

then, the equalities

$$u^{(4)} = u_{(4)}, \; v^{(4)} = v_{(4)}, \; D_{(8)} = D^1_{(7)} \cap [T^1_{(7)} \leqslant \min(A_1, A_2)] \in W_{(4)}$$

are valid. We introduce the notation

$$A_3 = \min[A_1, A_2, t_1(\alpha)]$$

$$D_{(9)}(w) = D^1_{(7)} \cap [T^1_{(7)} \leqslant A_3]$$

$$D_{(10)}(w) = D^{(1)} \setminus (D_{(9)} \cup D^{(2)})$$

$v_{(10)}(w) = +1$ for $w \in D_{(10)}(w)$

8.9. The relations

$$u_{(9)} = u^{(4)} = u^{*}, \quad v_{(9)} = v^{(4)} = +1 = v^{\circ}, \quad D_{(9)}(w) \in W_0(w)$$

are valid. The most difficult part of the proof of assertion 8.9 is the proof of the following property of the pair $[u \neq u^{(4)},\ v = v_{(10)}]$:

$$T^1_{(7)}(g_1) < T^1_{(7)}(g_2)$$

where g_1 is the point where the trajectory goes onto the common boundary $G_{(9)}$ of regions $D_{(9)}$ and $D_{(10)}$, while g_2 is the point where the trajectory returns to this boundary.

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NECESSARY OPTIMALITY CONDITIONS IN A LINEAR PURSUIT PROBLEM

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Necessary conditions are presented for the optimality of a certain guaranteed time (upper layer time [1]) for a large class of pursuit problems. Sufficient conditions of a general form have been cited in [1-5] and in a number of other papers for the possibility of terminating the pursuit at a specified time and the guarantee time effectively computed. Sufficient optimality conditions for guarantee times have been discussed in [6-8].

1. Suppose that a linear pursuit problem in an n-dimensional Euclidean space R is described:

a) by linear vector differential equations

$$\dot{z} = Cz - u + v \tag{1}$$

where C is a constant nth-order square matrix, $u = u(t) \in P$ and $v = v(t) \in Q$ are vector-valued functions, measurable for $t \geqslant 0$, called the controls of the players (the pursuer and pursued respectively); $P \subset R$ and $Q \subset R$ are convex compacta;

b) by a terminal set M representable in the form $M = M_0 + W_0$, where M_0 is a linear subspace of space R, and W_0 is some compact convex set in a subspace L